

ON THE CONVERGENCE OF POISSON INTEGRALS

BY

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1. **Introduction.** The object of this paper is to extend two partial results, one positive and one negative, concerning the a.e. convergence of Poisson integrals on generalized half-planes equivalent to bounded symmetric domains. These results involve the distinction between “restricted” and “unrestricted” convergence, which already arose in the case of domains which are equivalent to product of half-planes (for this case see e.g. [Z, Chapter 17]). For these special domains there is restricted convergence for L^p , $p \geq 1$, and unrestricted convergence for L^p , $p > 1$. The L^1 result of restricted convergence for various other tube domains was obtained more recently in [W₁] and [S]. The techniques set forth in [W₁] and [S] provide the starting point of our treatment here.

Turning to the case of the general bounded symmetric domains, the positive results of [W₁] and [W₂] show that the Poisson integral of a function f on one of the domains in question converges *restrictedly* to f at a.e. point of the boundary if $f \in L^p$, $p > 1$. It is demonstrated below that the condition $f \in L^1$ is also sufficient for restricted a.e. convergence, and that the Poisson integral of a measure has its Radon-Nikodym derivative as a restricted a.e. limit.

On the other hand, it was essentially shown in [SWW] that for domains of the above type, there exists $p_0 > 1$ such that every L^p class, $p < p_0$, contains a function whose Poisson integral has ∞ as an *unrestricted* supremum at a.e. point of the boundary.

The more complete result proved here is that p_0 can be taken to be ∞ in every case, except rank one; and that there exists $f \in L^\infty$ with a Poisson integral which at a.e. point of the boundary fails to converge unrestrictedly to f .

§2 is devoted to a few propositions of a more general nature which are necessary for the proof of the positive result in §3. The proof of the negative result is contained in §4.

2. **Real variable preliminaries.** The results given here are necessary for estimates of the behavior of maximal averages over classes of sets which arise in §3.

Note that the left Haar measure of a (measurable) subset E of a locally compact group is denoted by $|E|$.

The first auxiliary result, a covering theorem, generalizes results of Wiener [W] and Zygmund [Z, Chapter 17], and is proved by their method. See also [S].

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LEMMA 2.1. *Let G be a locally compact group and P a compact subset of G of positive measure. Suppose that $\{\alpha_t\}$, $t > 0$, is a multiplicative 1-parameter group of automorphisms of G such that $\alpha_t P \subset \alpha_{t'} P$ if $t < t'$. Suppose further that to every point x of a measurable subset E of G there is associated $t(x) > 0$ in such a way that*

$$\sup_{x \in E} |\alpha_{t(x)} P| < \infty.$$

Then there is a sequence $\{x_j\}$ of points of E such that

- (i) *the translates $x_j \cdot \alpha_{t(x_j)} P$ are disjoint;*
- (ii) $\sum_j |\alpha_{t(x_j)} P| \geq (CM_P)^{-1} |E|$, *where C depends only on G , and $M_P = |PP^{-1}P|/|P|$.*

Proof. The proof is by induction. We write $P_j = \alpha_{t(x_j)} P$, $R_j = x_j \cdot P_j$. To begin, choose x_0 such that $t(x_0) = T_0 = \sup_{x \in E} t(x)$. (We assume temporarily, here and below, that sups are attained.) Set $\bar{R}_0 = \bigcup \{x \cdot P_0 : x \cdot P_0 \cap R_0 \neq \emptyset\}$.

Now let $T_1 = \sup \{t(x) : x \notin \bar{R}_0\}$, choose $x_1 \notin \bar{R}_0$ such that $t(x_1) = T_1$, and set $\bar{R}_1 = \bigcup \{x \cdot P_1 : x \cdot P_1 \cap R_1 \neq \emptyset\}$. Continuing this process, we obtain a sequence of sets, $R_1, R_2, \dots$, which are clearly disjoint, and whose measures are nonincreasing in the index. If $\sum_j |R_j| = \infty$, we are done. If not, then $|R_j| \rightarrow 0$, i.e., $t(x_j) \rightarrow 0$. It thus follows that if $x \in E$ is not contained in any $\bar{R}_j$, then $t(x) = 0$, which is impossible, and so the $\bar{R}_j$ cover E .

Finally, it is easy to see that $\bar{R}_j \subset x_j \cdot P_j P_j^{-1} P_j$ for all j . Also,

$$|P_j P_j^{-1} P_j|/|P_j| = |PP^{-1}P|/|P| = M_P,$$

and so

$$|E| \leq \sum_j |\bar{R}_j| \leq M_P \sum_j |R_j|.$$

Dropping the hypothesis that the sups are attained is done in the usual way, and adds only a constant factor to our estimate.

COROLLARY 2.2. *Suppose that G , $\{\alpha_t\}$ and P are as in 2.1. Let $f \in L^1(G)$ and define*

$$f^*(x) = \sup_{t>0} |\alpha_t P|^{-1} \int_{\alpha_t P} |f(xy)| dy.$$

Then for $s > 0$,

$$|E_s| = |\{x : f^*(x) > s\}| < CM_P \|f\|_1/s.$$

Proof. If $x \in E_s$, we can choose $t(x)$ such that

$$|\alpha_{t(x)} P| < s^{-1} \int_{\alpha_{t(x)} P} |f(xy)| dy \leq s^{-1} \|f\|_1.$$

Thus the full hypothesis of Lemma 2.1 is satisfied. Let $x_1, x_2, \dots$ be the sequence obtained in the conclusion of Lemma 2.1. Then

$$|E_s| \leq CM_P \sum_j |R_j| \leq CM_P \|f\|_1/s,$$

where we have used the fact that the R_j are disjoint.

We express a result like Corollary 2.2 by saying that the transformation $f \rightarrow f^*$ is of weak-type 1-1, with bound CM_p . Weak-type estimates are not subadditive. For instance, if $|\{x : g_j(x) > s\}| < 1/s$, $j=1, 2$, the most that can be said is that $|\{x : g_1(x) + g_2(x) > s\}| < 4/s$. There is, however, a positive result which is sufficient for our needs. A related result is also stated in [S].

LEMMA 2.3. *Suppose that for $j=1, 2, \dots$, $g_j(x)$ is a nonnegative function on a measure space for which $|\{x : g_j(x) > s\}| < 1/s$. Let $\{c_j\}$ be a sequence of positive numbers with $\sum_j c_j = 1$, and set $K = \sum_j c_j \log(1/c_j)$.*

Then $|\{x : \sum_j c_j g_j(x) > s\}| < 2(K+2)/s$.

Proof. Define

$$\begin{aligned} l_j(x) &= g_j(x), & g_j(x) < s/2, \\ &= 0, & \text{otherwise,} \\ u_j(x) &= g_j(x), & g_j(x) > s/2c_j, \\ &= 0, & \text{otherwise,} \\ m_j(x) &= g_j(x) - l_j(x) - u_j(x). \end{aligned}$$

Then $l(x) = \sum c_j l_j(x) < s/2$ everywhere, and $u(x) = \sum c_j u_j(x) = 0$ except on a set of measure $< 2/s$.

Setting $\lambda_j(y) = |\{x : g_j(x) > y\}|$, $y > 0$, and $m(x) = \sum c_j m_j(x)$, we have

$$\begin{aligned} \int m(x) dx &= \int \sum c_j m_j(x) = - \sum c_j \int_{s/2}^{s/2c_j} y d\lambda_j(y) \\ &= \sum c_j \left[\int_{s/2}^{s/2c_j} \lambda_j(y) dy - y \lambda_j(y) \right]_{s/2}^{s/2c_j} < \sum c_j \left[\int_{s/2}^{s/2c_j} y^{-1} dy + 1 \right] \\ &= K + 1. \end{aligned}$$

And, in particular, $|\{x : m(x) > s/2\}| < 2(K+1)/s$.

The precise estimate we need is a direct consequence of Lemma 2.3.

COROLLARY 2.4. *If $g_j \geq 0$, $j=1, \dots, N$, and $|\{x : g_j(x) > s\}| < A/s$, then*

$$\left| \left\{ x : N^{-1} \sum_{j=1}^N g_j(x) > s \right\} \right| \leq 2A(2 + \log N)/s.$$

3. The positive result. The proof of the positive result uses the estimates of $[W_1]$ and $[W_2]$ to dominate the Poisson integral of a function f by a certain sum of maximal averages of f . A discrete version of the method of rotations used in $[W_1]$ and $[W_2]$, together with Corollaries 2.2 and 2.4, is then applied to establish the behavior of these maximal averages. The remainder of the proof is routine.

Finally, the methods of the proof are used to extend the convergence theorem to Poisson integrals of measures.

We begin by describing the setting. A generalized half-plane is a space of the form

$$D = \{(z, w) \in V_1 \times V_2 : \operatorname{Im} z - \Phi(w, w) \in \Omega\},$$

where V_1 and V_2 are complex Euclidean spaces, $\Omega \subset \operatorname{Re} V_1$ is an open convex cone, and $\Phi: V_2 \times V_2 \rightarrow V_1$ is a hermitian bilinear form such that $\Phi(w, w) \in \overline{\Omega}$, $w \in V_2$. When D is equivalent to a bounded symmetric domain, the case we consider here, Ω is a domain of positivity, i.e., is homogeneous and symmetric, and Φ also has certain homogeneity and symmetry properties. If $V_2 = \{0\}$, then D is a tube domain.

The distinguished boundary of D (referred to below as the boundary) is

$$B = \{(z, w) : \operatorname{Im} z - \Phi(w, w) = 0\},$$

and B is identified with $\operatorname{Re} V_1 \times V_2$ by identifying $(z, w) = (\operatorname{Re} z + i\Phi(w, w), w)$ with the pair $[\operatorname{Re} z, w]$. This identification will be used throughout the paper.

There is a nilpotent Lie group $\mathcal{N}$ of affine transformations of D , which is also identifiable with $\operatorname{Re} V_1 \times V_2$. The action of $\mathcal{N}$ is given by

$$[x, w] \cdot (z_0, w_0) = (z_0 + x + 2i\Phi(w_0, w) + i\Phi(w, w), w_0 + w),$$

and the group multiplication by

$$[x, w] \cdot [x', w'] = [x + x' + 2 \operatorname{Im} \Phi(w, w'), w + w'].$$

$\mathcal{N}$ is transitive on B , and we identify $g \in \mathcal{N}$ with $g \cdot 0 \in B$. The Euclidean measure on $\operatorname{Re} V_1 \times V_2$ induces a measure on B which is invariant under $\mathcal{N}$, and induces Haar measure on $\mathcal{N}$.

There is a Poisson kernel $P(u, \zeta)$ defined on $B \times D$, and the Poisson integral of a function f on B is

$$F(\zeta) = \int_B P(u, \zeta) f(u) du.$$

We are interested in the behavior of $F(\zeta)$ as ζ approaches a point $u \in B$. If $u = [x, w]$ and $y \in \Omega$, we define

$$uy = (x + iy + i\Phi(w, w), w) \in D.$$

If $\zeta = uy$, ζ is said to converge to u *restrictedly* if $y \rightarrow 0$ within a subcone of Ω . More generally, there is a type of convergence which extends the notion of restricted nontangential convergence. For $g = [x, w] \in \mathcal{N}$, we set $|g| = \max\{|x|, |w|^2\}$, and say that $\zeta \rightarrow u = g' \cdot 0$ *restrictedly and admissibly* if ζ stays within some

$$\Gamma_\alpha(u) = \{(g'g \cdot 0)_y : |g| < \alpha|y|\}$$

as $y \rightarrow 0$ within a subcone of Ω .

THEOREM 3.1. *Let D be a generalized half-plane equivalent to a bounded symmetric domain. Suppose that $f \in L^1(B)$ and that F is the Poisson integral of f .*

Then for a.e. $u \in B$, $F(\zeta) \rightarrow f(u)$ as $\zeta \rightarrow u$ restrictedly and admissibly.

We assume for now that D is irreducible and that $\zeta = uy$, where y is restricted to lie within a subcone Ω' of Ω . This will be shown to involve no loss of generality.

There exist maximal averages of f , sums of which dominate $F(uy)$, and we describe them now. Let $t, l, (j), (k)$ be, respectively, a positive number, a positive integer, an n -tuple and an m -tuple of nonnegative integers. There is defined in B (or $\mathcal{N}$) a subset $E_{(j)(k)l}^t$. (The exact definitions are given below.) We set

$$f_{(j)(k)l}^t(g) = |E_{(j)(k)l}^t|^{-1} \int_{E_{(j)(k)l}^t} |f(gg')| dg',$$

$$f_{(j)(k)l}^*(g) = \sup_{t>0} f_{(j)(k)l}^t(g).$$

LEMMA 3.2.

$$\sup_{y \in \Omega'} |F(g \cdot 0)_y| \leq A \sum_{(j)(k)} \sum_{l=1}^L 2^{-(|j|+2|k|)/2} f_{(j)(k)l}^*(g).$$

Proof. This is Lemma 6.8 of [W₁] and Proposition 3.4 of [W₂], and only a bare indication of the proof will be given here.

The idea is to break up the integral

$$F(g \cdot 0)_y = \int_{\mathcal{N}} f(g') P(g' \cdot 0, (g \cdot 0)_y) dg' = \int_{\mathcal{N}} f(gg') P(g' \cdot 0, (iy, 0)) dg'$$

by breaking up $\mathcal{N}$ into pieces $H_{(j)(k)l}^{|y|}$ which are contained in the $E_{(j)(k)l}^{|y|}$. It is found that

$$[\sup \{P(g' \cdot 0, (iy, 0)) : g' \in H_{(j)(k)l}^{|y|}\}] |E_{(j)(k)l}^{|y|}| \leq A 2^{-(|j|+2|k|)/2}$$

and this leads to the inequality in Lemma 3.2.

The reason for the exact form of the $E_{(j)(k)l}^t$, which will be given below, can be seen in a special case, when D is the tube domain over the cone Ω_n of positive definite $n \times n$ real symmetric matrices; in this case, $\mathcal{N}$ and B are realized as the space of all real symmetric $n \times n$ matrices and

$$P(g' \cdot 0, (iy, 0)) = c_n \{\det y / |\det (g' + iy)|^2\}^{(n+1)/2}.$$

Moreover, if the eigenvalues of g' are $r_1, \dots, r_n$ and if $y \in \Omega'_n \subset \Omega_n$, we have

$$P(g' \cdot 0, (iy, 0)) \leq a P(g' \cdot 0, (i|y|I, 0)) = ac_n \left\{ \frac{|y|^n}{(|y|^2 + r_1^2) \cdots (|y|^2 + r_n^2)} \right\}^{(n+1)/2}$$

and this naturally leads to a decomposition of $\mathcal{N}$ according to the size of eigenvalues. In fact, in this case we have

$$E_{(j)(k)l}^t = E_{(j)}^t = \{k^{-1}d(r_1, \dots, r_n)k : k \in U(n), |r_i| \leq 2^{j_i}t\},$$

where $d(r_1, \dots, r_n)$ is the diagonal matrix whose entries are the r_i .

The principal result of this section is a weak-type estimate on the $f_{(j)(k)l}^*$. From this result and Lemma 3.2, it will be easy to deduce that the transformation $f(u) \rightarrow \sup_{y \in \Omega'} |F(uy)|$ is of weak-type 1-1, and then to prove the main theorem.

LEMMA 3.3.

$$|\{g : f_{(j)(k)l}^*(g) > s\}| < C(1 + |j| + |k|) \|f\|_1/s.$$

The proof is involved, especially in the case of non-tube-type domains. In $[W_1]$ and $[W_2]$, it was shown that the f^* can be dominated by the integral over a compact group of maximal averages on rectangular sets in a Euclidean space. Known results about these, together with the integral Minkowski inequality, led to norm inequalities of the type $\|f^*\|_p \leq C \|f\|_p$, $p > 1$.

There is no similar integral inequality for weak-type norms, so we dominate the f^* by sums of maximal averages on rectangular sets in $\mathcal{N}$. If $\mathcal{N}$, as a group, is Euclidean (the tube domain case), the behavior of these maximal functions is known; in the more general case, we appeal to Corollary 2.2. An application of Corollary 2.4 then yields the desired inequality.

Proof of Lemma 3.3 for tube domains. D is a tube domain if $V_2 = \{0\}$. We use the fact that D can be imbedded in the complexification of a simple compact real Jordan algebra $\mathfrak{A}$. There is a decomposition $e = e_1 + \cdots + e_n$ of the identity of $\mathfrak{A}$ into orthogonal idempotents and a compact group K of $\mathfrak{A}$ automorphisms of $\mathfrak{A}$ such that for a.e. $x \in \mathfrak{A}$, there exists $k \in K$ such that $k(x) = \sum r_i e_i$, where the r_i are real. (This corresponds to the usual diagonalization in the case of matrix domains.) For details about $\mathfrak{A}$, here and below, refer to $[W_1, \S 5]$.

In this case, $\mathcal{N}$ is merely the algebra $\mathfrak{A}$, which is a Euclidean space. If we write

$$d(r) = d(r_1, \dots, r_n) = \sum r_i e_i,$$

and define

$$R_{(j)}^t = \{r : |r_i| \leq 2^t t_i\},$$

then the $E_{(j)(k)l}^t$ reduce to

$$E_{(j)}^t = \{x = k(d(r)) : r \in R_{(j)}^t\}$$

and

$$f_{(j)}^*(x) = \sup_{t>0} |E_{(j)}^t|^{-1} \int_{E_{(j)}^t} |f(x+x')| dx'.$$

We wish to express $f_{(j)}^*$ as a sum of maximal averages whose behavior is known. This is done by covering the group K with N pieces, $K_1, \dots, K_N$, each having measure γ/N , $\gamma \geq 1$, and setting

$${}_p E_{(j)}^t = \{x = k(d(r)) : r \in R_{(j)}^t, k \in K_p\}, \quad p = 1, \dots, N,$$

$${}_p f_{(j)}^* = \sup_{t>0} |{}_p E_{(j)}^t|^{-1} \int_{{}_p E_{(j)}^t} |f(x+x')| dx'.$$

It is then clear that

$$(1) \quad f_{(j)}^*(x) \leq \gamma N^{-1} \sum_{p=1}^N p f_{(j)}^*(x).$$

We show that such a covering of K exists which satisfies:

- (i) $|\{x : p f_{(j)}^*(x) > s\}| < A \|f\|_1/s,$
- (ii) $N \leq 2^{1/j} a,$

where, here as below, unidentified letters refer to constants which depend at most on the domain D and which may take on different values in different appearances. The properties (i) and (ii), taken together with (1) and Corollary 2.4, give the proof of the lemma in the tube domain case.

An inequality like (i) is known to hold if the averaging takes place over rectangular sets in Euclidean space which are the dilation of a fixed rectangle, with the constant A independent of this fixed rectangle. (This is also a trivial consequence of Corollary 2.2.) Therefore it is enough to find a covering which satisfies (ii) and (i') Every $p E_{(j)}^t$ can be covered by a rectangular set $p P_{(j)}^t$ in such a way that

$$|p P_{(j)}^t|/|p E_{(j)}^t| < b.$$

The covering $K_1, \dots, K_N$ is obtained as follows. Let M be the subgroup of K acting trivially on diagonal elements, and $\mathfrak{k}$ and $\mathfrak{m}$ be the Lie algebras of K and M . Then $K_1 = \exp(B_{(j)} + \mathfrak{m})$, where $B_{(j)}$ is a ball in $\mathfrak{k} - \mathfrak{m}$ whose radius ρ depends on (j) , and $K_2, \dots, K_N$ are sufficiently many translates of K_1 .⁽²⁾

By homogeneity, it is enough to show that (i') is satisfied for $p=1$ when ρ is sufficiently small ($\rho = a 2^{-1/j}$), and then to show that (ii) holds for this value of ρ . (For a special case, see [W₁, Theorem 4.2].)

The proof requires explicit knowledge of the structure of simple compact Jordan algebras. It is known that $\mathfrak{A}$ has an orthogonal basis of the form $\{e_i, s_{ik}^\lambda\}$, $i < k$, $i, k = 1, \dots, n$, $\lambda = 1, \dots, \chi$. (For details here and below, see [W₁, §5]. In the case of the matrix domains, the basis is the obvious one, and $\chi = 1, 2, 4$ or 8 , respectively, in the case of real, complex, quaternionic and Cayley number matrices.) Notice that $n_1 = \dim \mathfrak{A} = n[1 + \chi(n-1)/2]$.

Also, $\mathfrak{k} - \mathfrak{m}$ has a basis of the form $\{T_{ik}^\lambda\}$, where $T_{ik}^\lambda = [L(e_i - e_k), L(s_{ik}^\lambda)]$, L is left multiplication in $\mathfrak{A}$, and $[T, T'] = TT' - T'T$. In particular, $\dim \mathfrak{k} - \mathfrak{m} = n_1 - n$. Computation reveals that if $T = \sum a_{ik}^\lambda T_{ik}^\lambda$, then

$$(2) \quad T\left(\sum_l r_l e_l\right) = \sum_{i,k,\lambda} a_{ik}^\lambda \left(\frac{r_i - r_k}{2}\right) s_{ik}^\lambda,$$

$$(3) \quad T^2\left(\sum_l r_l e_l\right) = \sum_{i,k,\lambda} a_{ik}^\lambda \left\{ a_{ik}^\lambda \left(\frac{r_i - r_k}{2}\right) (e_i - e_k) + \sum_{l,\mu} a_{kl}^\mu \left(\frac{r_k - r_l}{2}\right) \left(\sum_v A_v s_{il}^v\right) + \sum_{m,\mu} a_{im}^\mu \left(\frac{r_i - r_m}{2}\right) \left(\sum_v A_v s_{km}^v\right) \right\},$$

where the A_v are structural constants.

⁽²⁾ $\mathfrak{k} - \mathfrak{m}$ denotes the orthogonal complement of $\mathfrak{m}$ in $\mathfrak{k}$ with respect to the Killing form.

Now the volume of all of $E_{(j)}^t = \{k(d(r)) : r \in R_{(j)}^t, k \in K\}$ is obtained by integrating the Jacobian $\Delta(r) = c \prod_{i < k} |r_i - r_k|^z$ over $R_{(j)}^t$. It turns out [W₁, Lemma 6.7] (but is not clear a priori) that $|E_{(j)}^t|$ can be estimated by merely multiplying together the largest possible values of the r_i and the greatest possible contribution of T to the s_{ik}^λ coordinates in (2) as the a_{ik}^λ range in absolute value from 0 to 1. Specifically, assuming that $j_1 \geq \dots \geq j_n$, we have

$$(4) \quad |E_{(j)}^t| \sim (2^{j_1 + \dots + j_n}) (2^{z[(n-1)j_1 + \dots + j_{n-1}]} t^{n_1},$$

$$(5) \quad |{}_1E_{(j)}^t| = |E_{(j)}^t| |K_1| \sim |E_{(j)}^t| \rho^{n_1 - n}.$$

($F_1(\cdot) \sim F_2(\cdot)$) means that $aF_1(\cdot) \leq F_2(\cdot) \leq bF_1(\cdot)$, where a and b are independent of the arguments.)

We find the rectangle ${}_1P_{(j)}^t$ which will cover ${}_1E_{(j)}^t$ by determining the greatest possible values of the coordinates of the points in ${}_1E_{(j)}^t$. Now ${}_1E_{(j)}^t$ consists of all $\exp T(d(r))$ such that $|r_i| \leq 2^{j_i}t$ and $T \in B_{(j)}$, i.e., $T = \sum a_{ik}^\lambda T_{ik}^\lambda$ with $\sum (a_{ik}^\lambda)^2 < \rho^2$. Referring to (2) and (3), we find that the greatest possible values of the e_i and s_{ik}^λ ($i < k$) coordinates in

$$\{(1 + T + T^2/2)(d(r)) : |r_i| \leq 2^{j_i}t, T \in B_{(j)}\}$$

are, respectively, multiples of

$$\sigma_i = (2^{j_i} + \rho^2 2^{j_1})t,$$

and

$$\sigma_{ik} = (2^{j_i} + \rho^2 2^{j_1})\rho t.$$

Moreover, higher powers of T only add terms of higher order in ρ and can be ignored, so the rectangular set ${}_1P_{(j)}^t$ can be taken to have sides whose lengths are multiples of σ_i and σ_{ik} . In particular,

$$(6) \quad |{}_1P_{(j)}^t| \sim \prod_{i=1}^n \sigma_i \prod_{i < k} (\sigma_{ik})^x \sim \prod_{i=1}^n (2^{j_i} + \rho^2 2^{j_1}) \prod_{i=1}^{n-1} (2^{j_i} + \rho^2 2^{j_1})^{[n-i]x} \rho^{n_1 - n} t^{n_1}.$$

Comparing (4), (5) and (6), we see that (i') holds if $\rho \leq \alpha 2^{-j_1}$, and certainly if $\rho = \alpha 2^{-|j|}$.

Finally, we show that (ii) holds, i.e., that K can be covered by $2^{|j|q} a = \rho^{-q} a'$ translates of K_1 , where $q = \dim(\mathfrak{k} - \mathfrak{m}) = n_1 - n$. We do this by appealing to a simple fact about compact Lie groups. (The application of this fact to a quotient space presents no problems.)

LEMMA 3.4. *Let K be a compact Lie group of dimension l with Lie algebra $\mathfrak{k}$. Let B_ρ be the ball about the origin in $\mathfrak{k}$ with radius ρ and let $K_\rho = \exp B_\rho$.*

Then there is a constant M , independent of ρ , such that K can be covered by fewer than $M\rho^{-l}$ translates of K_ρ .

Proof. Let $\bar{B}$ be the ball of radius 1 about the origin in $\mathfrak{k}$, and let $\bar{K} = \exp \bar{B}$. It is clearly enough to prove the lemma for $\bar{K}$ instead of K . Let M' be a number to

be determined, and $M = (M' \sqrt{l})^l$; we can cover $\bar{B}$ with $M\rho^{-l}$ balls, $\{B_j\}$, of radius ρ/M' . Denote by Y_j the center of B_j , and let $x_j = \exp Y_j$. We show that for M' (and thus M) sufficiently large, $\exp B_j \subset K_\rho \cdot x_j$; this will complete the proof, since the B_j cover $\bar{B}$.

What we must show is that if $|X| < \rho/M'$, then

$$\exp(Y_j + X) = \exp X' \exp Y_j, \quad X' \in B_\rho,$$

i.e.,

$$|\log [\exp(Y_j + X) \exp(-Y_j)]| < \rho.$$

But the function $F(Y, X, t) = \log [\exp(Y + tX) \exp(-Y)]$ is analytic in Y, X, t , and $F(Y, X, 0) \equiv 0$, while the analytic function $\partial F(Y, X, t)/\partial t$ is bounded for $Y \in \bar{B}$, $|X| = 1$ and $0 \leq t \leq 1$, and so the lemma is proved.

In summary, we have established (i') and (ii) with $\rho = \alpha 2^{-|j|}$. Clearly $|K_1| \leq \gamma' 2^{-|j|} \leq \gamma/N$, and so we have proved 3.3 in the case of tube domains.

Proof of Lemma 3.3 for type I domains. The nontube domains in which we are interested fall into two large classes, with the exception of an exceptional domain of dimension 16. There is a domain of type I for every pair of integers n, m , with $n > 0, m \geq 0$; in the realization we consider, V_1 is the complexification of the (real) vector space of $n \times n$ complex hermitian symmetric matrices, Ω is the cone of positive definite matrices, V_2 is the space of $n \times m$ complex matrices, and $\Phi(w, \omega) = w\omega^*$.

If $x \in \text{Re } V_1$, then $x = k^{-1}d(r)k$, $k \in U(n)$, where $d(r) = d(r_1, \dots, r_n)$ is the diagonal matrix whose entries are the r_i . And if $w \in V_2$, $w = u\tilde{d}(s)v$, $u \in U(n)$, $v \in U(m)$, where $\tilde{d}(s)$ is the diagonal-type $n \times m$ matrix whose entries are $s_1, \dots, s_m$ (resp. $s_1, \dots, s_n$) if $m \leq n$ (resp. $n \leq m$). We assume for convenience that $m \leq n$, the case $n \leq m$ is dealt with similarly.

For every n - and m -tuple, $(j), (k)$, there is defined a decreasing family of neighborhoods of the identity in $U(n)$,

$$U(n) = N_{(j)(k)1} \supset \dots \supset N_{(j)(k)L}.$$

These neighborhoods are defined as exponential images of balls about the origin in $U(n)$, and $N_{(j)(k)L}$ may be empty for $L' \leq L \leq L$. Also, the measure of the smallest nonempty N has measure greater than $2^{-q'(|j| + |k|)} a'$. (See [W₂, §3].)

Recalling the definition of $R_{(j)}^t \subset E_n$, and defining $S_{(k)}^t \subset E_m$ similarly, we define

$$E_{(j)(k)l}^t = \{[x, w] = [k^{-1}d(r)k, u\tilde{d}(s)v] : r \in R_{(j)}^t, s \in S_{(k)}^{t^{1/2}}, ku \in N_{(j)(k)l}\}.$$

The reason for this complicated definition may be seen by considering the formula for the Poisson kernel in this case; specifically,

$$P([x, w], (iy, 0)) = C_{nm} \{ \det y / |\det(x + i[y + w\omega^*])|^2 \}^{n+m}.$$

And, for example, if $x = k^{-1}d(2^j|y|, 0, \dots, 0)k$, $w = u\tilde{d}(2^k|y|^{1/2}, 0, \dots, 0)v$, the value of $P([x, w], (iy, 0))$ ranges from

$$C_{nm}[(2^{2j} + (2^{2k} + 1)^2)|y|^n]^{-(n+m)}$$

to

$$C_{nm}[(2^{2j} + 1)(2^{2k} + 1)^2|y|^n]^{-(n+m)},$$

depending on ku . We repeat for emphasis the definition

$$f_{(j)(k)l}^*(g) = \sup_{t>0} |E_{(j)(k)l}^t|^{-1} \int_{E_{(j)(k)l}^t} |f(gg')| dg'.$$

The idea of the proof of Lemma 3.3 here is the same as in the tube domain case, that is, we express each $f_{(j)(k)l}^*$ as a sum of maximal averages over rectangular sets. The first step of the proof consists of verifying the analogs of (i') and (ii) for a covering of $\tilde{K} = U(n) \times U(n) \times U(m)$. In this case, however, we must prove more, since the behavior of maximal averages over rectangular sets in $\mathcal{N}$ is not known, but must be determined with the aid of Corollary 2.2. ($\mathcal{N}$ is not a Euclidean space as a group, but we have for it Euclidean coordinates, so the notion of "rectangular set" is well defined.)

Fixing (j) , (k) and l , a covering $\tilde{K}_1, \dots, \tilde{K}_N$ of $\tilde{K}$ will be obtained by taking products of coverings of the factors of $\tilde{K}$. Defining ${}_pE_{(j)(k)l}^t$, $p=1, \dots, N$, as in the tube domain case, by restricting (k, u, v) to $\tilde{K}_p$, we find a covering which satisfies:

(I') Every ${}_pE_{(j)(k)l}^t$ can be covered by a rectangular set ${}_pP_{(j)(k)l}^t$ in such a way that $|{}_pP_{(j)(k)l}^t|/|{}_pE_{(j)(k)l}^t| < b$.

(II) $N \leq a2^{q(|j|+|k|)}$.

We consider first the case $l=1$, deferring consideration of the complications caused when $l>1$ by the restriction $ku \in N_{(j)(k)l}$ in the definition of $E_{(j)(k)l}^t$. Specifically, our covering of $\tilde{K}$ will be a product of coverings $\{K_j\}, \{K'_j\}, \{K''_j\}$ of $U(n)$, $U(n)$ and $U(m)$, respectively. By homogeneity, it is enough to establish (I') for $\tilde{K}_1 = K_1 \times K'_1 \times K''_1$, a sufficiently small neighborhood of the identity in $\tilde{K}$, and to show that $\tilde{K}$ can be covered by N translates of $\tilde{K}_1$, where N satisfies (II).

Suppressing the indices (j) , (k) , l , t , we write

$$\begin{aligned} {}_1F &= \{k^{-1}d(r)k \in \text{Re } V_1 : r \in R_{(j)}^t, k \in K_1\}, \\ {}_1G &= \{u\tilde{d}(s)v \in V_2 : s \in S_{(k)}^{t^{1/2}}, (u, v) \in K'_1 \times K''_1\}. \end{aligned}$$

Notice that

$${}_1E_{(j)(k)l}^t = {}_1E = {}_1F \times {}_1G.$$

We must choose K_1, K'_1, K''_1 so that ${}_1E$ can be covered by a rectangular set ${}_1P = {}_1Q \times {}_1R$ in such a way that the inequality in (I') is satisfied.

The desired K_1 , i.e., one for which $|{}_1Q|/|{}_1F| < b_1$, has been found in the proof of Lemma 3.3 for tube domains, and $U(n)$ can be covered by fewer than $a_1 2^{q_1|j|}$ translates of this K_1 .

The desired K'_1 (resp. K''_1) is constructed by taking the exponential image of a ball plus a subspace in $u(n)$ (resp. $u(m)$). The situation is like that in the tube case. A lower bound on the volume of ${}_1G$ can be obtained by multiplying together the greatest possible sizes of the s_i coordinates in $S_{(k)}^{t^{1/2}}$ and the greatest possible sizes of all the other coordinates in $\{X\tilde{d}(s)Y\}$, where $s \in S_{(k)}^{t^{1/2}}$ and X and Y lie in balls in $u(n)$ and $u(m)$, ([W₂, §3]). And the volume of ${}_1R$, the rectangular set which covers ${}_1G$, can be bounded above by taking the product of the greatest possible sizes of all the coordinates in $\{(1+X+X^2/2)\tilde{d}(s)(1+Y+Y^2/2)\}$.

The detailed estimates for $|{}_1G|$ and $|{}_1R|$ are straightforward but unappealing; the answer is that for (I') to hold, the radii of the balls in $u(n)$ and $u(m)$ can each be taken to be $\alpha 2^{-2|k|}$.

By Lemma 3.4, $U(n)$ and $U(m)$ can be covered by fewer than $a_2 2^{q_2|k|}$ translates of K'_1 and K''_1 , respectively, and so $\tilde{K}$ can be covered by fewer than $2^{q(|j|+|k|)}a$ translates of $\tilde{K}_1$, and we have established (I') and (II) in the case $l=1$.

The situation when $l>1$, in which we must take into account the restriction $ku \in N_{(j)(k)l}$, is not much more difficult. We start by restricting the covering of $\tilde{K}$ by translates of $\tilde{K}_1$ which was found in the case $l=1$ to a covering of

$$\{(k, u, v) \in \tilde{K} : ku \in N_{(j)(k)l}\}.$$

Problems arise only with $\tilde{K}_p = K_i \times K'_i \times K''_i$ for which $K_i \cdot K'_i$ is not completely contained in $N_{(j)(k)l}$. (In this case, $|{}_pP_{(j)(k)l}^t| = |{}_pP_{(j)(k)1}^t|$, while $|{}_pE_{(j)(k)l}^t|$ may be smaller than $|{}_pE_{(j)(k)1}^t|$.) It is, however, enough that

$$(7) \quad |\{(k, u) \in K_i \times K'_i : ku \in N_{(j)(k)l}\}| / |K_i| |K'_i| > \frac{1}{2}.$$

Since the smallest $N_{(j)(k)l}$ is of measure greater than $2^{-q'(|j|+|k|)}a'$, a covering of the K_i and K'_i by smaller sets of the same form can be constructed which gives us (7) without violating (II), except for a possible change in the constants a and q .

With (I') and (II) established, we are faced with the final problem of showing that the transformation

$$f(g) \rightarrow {}_p\tilde{f}_{(j)(k)l}(g) = \sup_{t>0} |{}_pP_{(j)(k)l}^t|^{-1} \int_{{}_pP_{(j)(k)l}^t} |f(gg')| dg'$$

is of weak-type 1-1 with a bound independent of p , (j) , (k) and l .

The automorphisms $\alpha_t(x, w) = (tx, t^{1/2}w)$ and the subset ${}_pP_{(j)(k)l}^1$ of $\mathcal{N}$ are certainly of the form prescribed in the hypotheses of Lemma 2.2. And so to verify that the weak-type bound is uniform, it is enough to establish

(III) Let $P = {}_1P_{(j)(k)l}^1$; then $|PP^{-1}P|/|P| < C$.

Recall the formula for group multiplication on $\mathcal{N}$:

$$[x, w] \cdot [\xi, \omega] = [x + x' + 2 \operatorname{Im} w\omega^*, w + \omega].$$

In the absence of the $\operatorname{Im} w\omega^*$ term, we would have $|PP^{-1}P|/|P| = 3^\nu$, where $\nu = n^2 + 2nm = \dim \mathcal{N}$. Since $P = Q \times R$, where Q and R are rectangular sets in $\operatorname{Re} V_1$

and V_2 , respectively, we can establish (III) by showing that if $w, \omega \in R$, then $|\operatorname{Im} w\omega^*|$ is no greater than the size of the smallest side of Q . And we saw in the proof of Lemma 3.3 for tube domains that the smallest side of $Q = {}_p Q_{(j)}^1$ is at least a fixed multiple of 1.

Moreover, $\operatorname{Im} w\omega^* = (w\omega^* - \omega w^*)/2i$, so

$$|\operatorname{Im} (uvw)(u\omega v)^*| = |\operatorname{Im} (u[w\omega^*]u^{-1})| = |\operatorname{Im} (w\omega^*)|,$$

and it is thus sufficient to prove (III) in the case $p=1$, i.e., at the origin of $\tilde{K}$.

If $(w_{ij}), (\omega_{ij})$ are the representations of w and ω as $n \times m$ complex matrices, then $\operatorname{Im} (w\omega^*) = (\eta_{ij})/2i$, where

$$(8) \quad \eta_{ij} = \sum_{k=1}^m (w_{ik}\bar{\omega}_{jk} - \omega_{ik}\bar{w}_{jk}).$$

Let $w_{ij} = w'_{ij} + iw''_{ij}$, $\omega_{ij} = \omega'_{ij} + i\omega''_{ij}$, and notice in particular that the sum in (8) contains no terms of the form $w'_{ii}\omega'_{jj}$ or $w'_{jj}\omega'_{ii}$. If $\rho = \alpha 2^{-2|k|}$ is the radius of the balls in $u(n)$ and $u(m)$ which determine K'_1 and K''_1 , then except for w'_{jj} , w'_{ii} , ω'_{jj} , ω'_{ii} , whose sizes are bounded by multiples of $2^{|k|}$, all of the w' , w'' , ω' , ω'' have sizes bounded by multiples of $2^{|k|}\rho = \alpha 2^{-|k|}$. Therefore $|\eta_{ij}| \leq C$, and we have established (III) and so completed the proof of Lemma 3.3 for type I domains.

Proof of Lemma 3.3 for type III b domains. Domains of type III b, of which there is one for each positive integer n , can be realized by letting V_1 be the complexification of the space of quaternionic hermitian symmetric matrices, Ω the cone of positive definite matrices, V_2 the space of n -long quaternionic column vectors, and $\Phi(w, \omega) = w\omega^*$.

The proof follows the exact lines of the proof in the type I case, and all details are omitted.

Proof of Lemma 3.3 for the exceptional domain. The exceptional domain D can be realized by taking V_1 and V_2 to be the ordinary 8-dimensional Euclidean space over the complex numbers and Ω to be the forward light cone in real 8-space. If we write $\xi = (x_2, \dots, x_8)$, then $\Omega = \{(x_1, \xi) : x_1 > |\xi|\}$.

The sets $E_{(j)(k)l}^t$ depend only on $(j) = (j_1, j_2)$ and $(k) = k$, and are given by:

$$E_{(j)k}^t = \{[(x_1, \xi), w] : |x_1| + |\xi| \leq 2^{j_1}t, | |x_1| - |\xi| | \leq 2^{j_2}t, |w| \leq 2^k t^{1/2}\}.$$

The argument used in the type I case can be repeated once more. The properties (I') and (II) are immediate. In V_2 we need only cover the sphere of radius $2^k t^{1/2}$ by the obvious cube, while in $\operatorname{Re} V_1$, we can appeal to the proof of Lemma 3.3 for tube domains to find a covering of $K = SO(8)$ which leads to (I') such that $N \leq a 2^{q|j|}$.

There remains only the verification of (III). For this we use the fact (communicated to us by A. Koranyi) that the place of $U(n) \times U(m)$ in the type I case is taken in the exceptional case by the group $L = SO(7) \times SO(2)$. In particular, there is a subspace

$V' \subset V_2$ of real dimension 4 whose image under the action of L is all of V_2 . L acts also on V_1 , and if $u \in L$,

$$|\Phi(uw, u\omega)| = |u\Phi(w, \omega)| = |\Phi(w, \omega)|,$$

and so it is again sufficient to consider the situation at the identity in L .

We have a rectangular set $Q \subset \text{Re } V_1$, whose smallest side is at least a multiple of 1. We take a neighborhood L_1 of the origin in L , determined by a ball of radius ρ in the Lie algebra of L , a rectangular set S in V' , none of whose sides exceed a multiple of 2^k , and let G be the set obtained by the action of L_1 on S . Then G can be covered by a rectangular set R such that the sides of R are of length less than a multiple of $2^k\rho$, except for the sides corresponding to V' , which are of length less than a multiple of 2^k . Since $\text{Im } \Phi(w', \omega') = 0$, $w', \omega' \in V'$, it follows as in the matrix case that if $w, \omega \in R$, then $|\Phi(w, \omega)| < C2^{2k}\rho$. Choosing $\rho = 2^{-2k\alpha}$, and covering L by translates of L_1 , we have (III) and still have (II). A final application of Corollaries 2.2 and 2.4 completes the proof of Lemma 3.3 for the exceptional domain, and thus for all irreducible domains equivalent to bounded symmetric domains.

COROLLARY 3.5. *Let*

$$f^{**}(g) = \sup_{y \in \Omega'} |F(g \cdot 0)_y|.$$

Then

$$|\{g : f^{**}(g) > s\}| < B\|f\|_{1/s},$$

where B depends only on Ω' .

Proof. Define

$$f_*(g) = \sup_{(j)(k)l} 2^{-(|j|+2|k|)/4} f_{(j)(k)l}^*(g).$$

Then

$$\begin{aligned} |\{g : f_*(g) > s\}| &\leq \sum_{(j)(k)l} |\{g : f_{(j)(k)l}^*(g) > 2^{(|j|+2|k|)/4}s\}| \\ &\leq C \sum_{(j)(k)l} [1 + |j| + |k|] \|f\|_{1/2^{(|j|+2|k|)/4}s} = C' \|f\|_{1/s}, \end{aligned}$$

where we have applied Lemma 3.3.

On the other hand, it follows from Lemma 3.2 that

$$f^{**} \leq A \sum_{(j)(k)l} 2^{-(|j|+2|k|)/2} f_{(j)(k)l}^* \leq A \sum_{(j)(k)l} 2^{-(|j|+2|k|)/4} f_* = A' f_*,$$

where A depends only on Ω' . This proves Corollary 3.5.

Proof of Theorem 3.1. First of all, the restriction that $\zeta = u_y$ can be lifted, since $[W_2, \S 3]$ $P(\cdot, \zeta) \leq C_\alpha P(\cdot, u_y)$ if $\zeta = u'_y \in \Gamma_\alpha(u)$. And the case when D is a product of irreducible domains can be handled in a standard way, ([Z, Chapter 17], $[W_1, \S 7]$). In each case, the conclusion of Corollary 3.5 still holds.

Finally, we write f as the sum of a continuous function of compact support (whose Poisson integral converges everywhere) and a function whose L^1 norm is small. We then apply Corollary 3.5 to show that the set where F does not converge to f has arbitrarily small measure.

THEOREM 3.6. *Suppose that μ is a finite Borel measure on B whose Radon-Nikodym derivative with respect to Euclidean (Lebesgue) measure on B is $f(u)$. Let*

$$F(\zeta) = \int_B P(u, \zeta) d\mu(u).$$

Then for a.e. $u \in B$, $F(\zeta) \rightarrow f(u)$ as $\zeta \rightarrow u$ restrictedly and admissibly.

Proof. By 3.1, it is enough to assume that μ is singular with respect to Lebesgue measure and to show that $F(\zeta) \rightarrow 0$ a.e. on B . We write $\mu = \lambda + \nu$, where $\|\lambda\| < \varepsilon$ and ν is supported in a closed subset N of B . It follows as in the proof of Theorem 3.1 (Corollary 3.5) that if F_λ is the Poisson integral of λ , then

$$\left| \left\{ u : \sup_y |F_\lambda(u_y)| > s \right\} \right| < A\varepsilon/s.$$

Letting ε be arbitrarily small, we are reduced to showing that for a.e. $u \in B$, $F_\nu(\zeta) \rightarrow 0$ as $\zeta \rightarrow u$.

Assume once more that $\zeta = u_y = (g \cdot 0)_y$. It follows as in Lemma 3.2 that

$$(9) \quad |F_\nu(g \cdot 0)_y| \leq A \sum_{l=1}^L \left\{ \sum_{|j|+2|k| < M} 2^{-(|j|+2|k|)/2} \nu_{(j)(k)l}^{[y]}(g) + \sum_{|j|+2|k| > M} 2^{-(|j|+2|k|)/2} \nu_{(j)(k)l}^*(g) \right\}$$

where

$$\begin{aligned} \nu_{(j)(k)l}^{[y]}(g) &= |E_{(j)(k)l}^{[y]}|^{-1} \int_{E_{(j)(k)l}^{[y]}} |dv|(gg'), \\ \nu_{(j)(k)l}^*(g) &= \sup_{t>0} \nu_{(j)(k)l}^t(g). \end{aligned}$$

Again appealing to Corollary 3.5, we see that the measure of the set on which the second term in (9) exceeds a given positive number is arbitrarily small if M is sufficiently large. On the other hand, since the support N of ν is closed, it is clear that each $\nu_{(j)(k)l}^{[y]}(g) \rightarrow 0$ as $y \rightarrow 0$ whenever $g \cdot 0 \notin N$. Since $|N|=0$, the proof is complete.

4. The negative result. In this section, as before, D is a generalized half-plane equivalent to a bounded symmetric domain, B is the distinguished boundary of D , and the Poisson integral of a function f on B is denoted by F .

It was shown in [JMZ] that if D is the Cartesian product of ordinary half-planes and $f \in L^p(B)$, $p > 1$, then $F(u_y) \rightarrow f(u)$ a.e. on B as $y \rightarrow 0$ *unrestrictedly*. (This was

proved in [JMZ] as a corollary to the *strong* a.e. differentiability of the indefinite integral of an L^p function, $p > 1$.)

On the other hand, it was shown in [SWW] that if D is the tube domain over the forward light cone in n -space (and, implicitly, if D is any irreducible domain of the type we are considering), then there exists $p_0 > 1$ such that if $p < p_0$, there is a function $f \in L^p(B)$ such that for a.e. $u \in B$, $\limsup F(u_y) = \infty$ as $y \rightarrow 0$ unrestrictedly.

The more complete negative result is

THEOREM 4.1. *Suppose that D is irreducible, and D is not equivalent to a hypersphere. (That is, D is not equivalent to a type I domain with $n=1$.) Then*

(a) *There exists $f \in L^\infty(B)$ such that for a.e. $u \in B$, $F(u_y) \nrightarrow f(u)$ as $y \rightarrow 0$ unrestrictedly.*

(b) *For every $p < \infty$, there exists $f \in L^p(B)$ such that for a.e. $u \in B$, $\limsup F(u_y) = \infty$ as $y \rightarrow 0$ unrestrictedly.*

REMARK. If D is equivalent to a hypersphere, then its Ω is merely a half-line, and unrestricted convergence is the same as restricted convergence. Therefore the positive result holds in that case.

The proof of (a) consists in essence of showing that for characteristic functions of sets, the convergence problem is equivalent to a differentiation problem, and then using a construction of Nikodym to provide a negative answer to the differentiation problem.

The proof of (b) follows from (a) and some general arguments.

Proof of Theorem 4.1(a). We consider first the tube domain case, in which the Poisson integral of a function is its convolution with the Poisson kernel P_y . D will again be considered as imbedded in the complexification of a Jordan algebra $\mathfrak{A}$.

The quadratic representation [W₁, §5] takes an element $a \in \mathfrak{A}$ into the linear transformation $Q(a): x \rightarrow 2a(ax) - a^2x$, and satisfies:

$$P_y(Q(y^{1/2})x) = [\det Q(y^{1/2})]^{-1}P_e(x), \quad y \in \Omega.$$

We make use of this in writing

$$\begin{aligned} (1) \quad F(u+iy) - f(u) &= \int_{\mathfrak{A}} [f(u-x) - f(u)]P_y(x) dx \\ &= \int_{\mathfrak{A}} [f(u - Q(y^{1/2})x) - f(u)]P_e(x) dx = \int_{Q_N} + \int_{\mathcal{C}_{Q_N}}, \end{aligned}$$

where Q_N is the cube of side N about the origin. The second integral is dominated by $2\|f\|_\infty \int_{\mathcal{C}_{Q_N}} P_e(x) dx$, which is arbitrarily small for sufficiently large N , and therefore $F(u+iy) \rightarrow f(u)$ as $y \rightarrow 0$ if

$$\int_{Q_N} |f(u - Q(y^{1/2})x) - f(u)|P_e(x) dx \leq C \int_{Q_N} |f(u - Q(y^{1/2})x) - f(u)| dx \rightarrow 0$$

as $y \rightarrow 0$.

Setting $E_N^y = \{x : Q(y^{1/2})^{-1}x \in Q_N\}$, the last integral is

$$C|Q_N| |E_N^y|^{-1} \int_{E_N^y} |f(u-x) - f(u)| dx,$$

and we see that for any $f \in L^\infty$, a.e. convergence of $F(u_y)$ to $f(u)$ follows from a.e. differentiation of the indefinite integral of f with respect to sets of the form $E^y = E_1^y$. (This is an adaptation of the argument used by Koranyi and Stein in [KS].)

On the other hand, if f is the characteristic function of a set, then the integrands in (1) are always of a single sign; also, $P_e(x)$ is bounded below if $|x| \in Q_1$. We therefore have

$$\begin{aligned} |F(u+iy) - f(u)| &\geq C' \int_{Q_1} |f(u - Q(y^{1/2})x) - f(u)| dx \\ &= C'' |E^y|^{-1} \int_{E^y} |f(u-x) - f(u)| dx. \end{aligned}$$

In particular, if a.e. differentiation fails for a characteristic function f , convergence of the Poisson integral fails for f .

As we shall see, the case of the (unique) irreducible tube domain D in 3-space is typical. This domain can be realized by letting $B = \mathfrak{A}$ be the Jordan algebra of 2×2 real symmetric matrices and Ω the cone of positive definite matrices. The multiplication in $\mathfrak{A}$ is given by $x \circ x' = \frac{1}{2}(xx' + x'x)$, and so $Q(y^{1/2})x = y^{1/2}xy^{1/2}$, and $E^y = \{y^{1/2}xy^{1/2} : x \in Q_1\}$. Suppose that y is of the form

$$d(y_1, y_2) = \begin{pmatrix} y_1 & 0 \\ 0 & y_2 \end{pmatrix}.$$

Then a simple computation shows that

$$(2) \quad E^y = \left\{ \begin{pmatrix} x_1 & x_3 \\ x_3 & x_2 \end{pmatrix} : |x_1| \leq y_1, |x_2| \leq y_2, |x_3| \leq (y_1 y_2)^{1/2} \right\}.$$

Moreover, $E^{k^{-1}yk} = k^{-1}E^y k$, and so the most general set E^y is a rotation of a set of the above form.

Bringing together everything that has been said so far, we see that for $f = \chi_H$ to fail to be the a.e. limit of its Poisson integral, it is sufficient that the indefinite integral of f does not have f as its derivative a.e. with respect to sets of the form E^y . Define the H -density of a point x with respect to the sets E^y to be

$$\lim_{y \rightarrow 0} \frac{|(x + E^y) \cap H|}{|E^y|}.$$

Then another sufficient condition for the failure of a.e. convergence of the Poisson integral of χ_H is that no a.e. point of H has H -density 1 with respect to $\{E^y\}$.

Notice now that by (2) and the sentence after it, we can make the statement

S: $\{E^y\}$ contains a class of rectangular parallelepipeds having one side arbitrarily larger than the other two and their long axes on the boundary of the cone Ω .

The existence of a set H , no point of which has H -density 1 with respect to the class of sets mentioned in **S**, will follow as a corollary to the next result, due to Nikodym [N].

THEOREM 4.2. *There exists a plane set H_0 which is contained in the unit square such that $|H_0|=1$ and every point $x \in H_0$ lies on an (infinitely long) line l_x whose intersection with H_0 is $\{x\}$.*

The consequence we want is

COROLLARY 4.3. *There exists a set $\tilde{H}$ contained in a unit cube in 3-space such that $|\tilde{H}|=1$ and every point $x \in \tilde{H}$ lies on a line L_x which is parallel to some ray on the boundary of Ω , and which has $\{x\}$ as its intersection with $\tilde{H}$.*

Proof. The boundary of the cone Ω is $\{(x_1, x_2, x_3) : x_3^2 = x_1 x_2, x_1 \geq 0, x_2 \geq 0\}$. It is a cone whose central axis is the ray ρ_0 , $x_1 = x_2 \geq 0, x_3 = 0$. Under the linear change of coordinates $x_3 = x'_3, x_1 = x'_1 + x'_2, x_2 = x'_1 - x'_2$ it becomes a circular cone in the (x'_1, x'_2, x'_3) space. Now let $\mathcal{P}$ be the plane passing through the origin and perpendicular to the central axis ρ_0 . Notice that every line in that plane, passing through the origin is the projection on $\mathcal{P}$ of a line lying in the boundary of Ω . Consider the plane set, given by Theorem 4.2 as lying in $\mathcal{P}$, and let $\tilde{H}$ be the Cartesian product of H_0 with the segment of unit length of the ray ρ_0 given by $1 \geq x_1 = x_2 \geq 0, x_3 = 0$. Because of Theorem 4.2, given any $x \in \tilde{H}$ there is a line l_x lying in the plane $\mathcal{P}$, which meets H_0 only at that point which is the projection of x on $\mathcal{P}$. Now let L_x be the line passing through x which is parallel to a line in the boundary of Ω , and so that the projection of L_x on $\mathcal{P}$ is l_x . Since every plane section of $\tilde{H}$, parallel with the plane $\mathcal{P}$, is identical with H_0 it is clear that $L_x \cap \tilde{H} = \{x\}$, and the corollary is proved.

Zygmund noted [N, p. 168] that the existence of the set H_0 above provides a negative answer to the question of the differentiability of the indefinite integral with respect to the class of all rectangles whose center is the origin. We make the same observation here, calling it

COROLLARY 4.4. *There is a subset H of 3-space such that $|H| > \frac{1}{2}$ and no point of H has H -density 1 with respect to the class named in statement **S**, and in particular, with respect to the class $\{E^v\}$.*

Proof. It is enough to let H be a closed subset of $\tilde{H}$ such that $|H| > \frac{1}{2}|\tilde{H}|$. Given $x \in H$, let L_x be the line through x as in Corollary 4.3. Then for any ε , it is possible to find a rectangular parallelepiped R_x (of the class $\{E^v\}$) whose center is at x and whose long axis lies on L_x and has length ε , such that $|R_x \cap H|/|R_x| < \frac{1}{2}$. In fact, the set λ of points on L_x whose distance from x is between $\varepsilon/4$ and $\varepsilon/2$ lies at a positive distance from H , and so can be covered by a parallelepiped R_x of the prescribed form whose cross-sections across λ do not meet H ; clearly,

$$|R_x \cap H|/|R_x| < \frac{1}{2}.$$

To summarize, if $f_0 = \chi_H$, then for every $u \in H$, $F_0(u_y) \rightarrow f_0(u)$ as $y \rightarrow 0$ unrestrictedly. It is easy now to find $f \in L^\infty(B)$ such that for a.e. $u \in B$, $F(u_y) \rightarrow f(u)$. Let $\{r_j\}$ be an enumeration of the points in B (identified with 3-space) whose coordinates are rational. If we denote by H_j the translate $H + r_j$, the desired function is $f = \sum_j 2^{-j} \chi_{H_j}$. To see this it is enough to show that the H_j cover a.e. point of B .

We show that every point of B is a point of (ordinary) $\bar{H}$ -density 1, where $\bar{H} = \bigcup H_j$, which is impossible if $\mathcal{C}\bar{H}$ has positive measure. Let h be a point of density of H ; then for all $\varepsilon > 0$, there exists $\delta > 0$ such that if Q is a cube about h with $|Q| < \delta$, then $|Q \cap H|/|Q| > 1 - \varepsilon$. Now suppose $b \in B$ and let Q' be a cube about b with $|Q'| < \delta$. Clearly there is an r_j such that $h + r_j$ is the center of a cube Q'' satisfying $Q'' \subset Q'$, $|Q''|/|Q'| > 1 - \varepsilon$. Finally,

$$|Q' \cap H|/|Q'| > (|Q'' \cap H|/|Q''|) \cdot (|Q''|/|Q'|) > 1 - 2\varepsilon,$$

and this completes the proof.

We conclude the proof of Theorem 4.1(a) by showing that the case of the irreducible domain in 3-space is typical. Any other irreducible tube domain (save the upper half-plane) has dimension greater than 3, and the argument above can be repeated. The failure of a.e. unrestricted convergence reduces to the existence of a set H , no point of which has H -density 1 with respect to a class of rectangular sets which have one side arbitrarily longer than all the others and their long axes on the boundary of a cone Ω . And the existence of such a set again follows from the theorem of Nikodym.

Finally, we turn to the nontube case. The quadratic representation $Q(y)$, $y \in \Omega$, can be extended from $\text{Re } V_1 (= \mathfrak{U})$ to all of $B (= \mathcal{N})$ and still satisfies

$$P(Q(y^{1/2})u, (iy, 0)) = [\det Q(y^{1/2})]^{-1} P(u, (ie, 0)).$$

Suppose now that $f = \chi_{H^*}$, $H^* \subset B$. Then, with the usual identification of B with $\mathcal{N}$ understood, we have

$$\begin{aligned} \Delta_y &= |F([x_0, w_0]_y) - f([x_0, w_0])| \\ &= \int_{\mathcal{N}} P([x, w], (iy, 0)) |f([x_0, w_0] \cdot [x, w]) - f([x_0, w_0])| dx dw \\ (3) \quad &= \int_{\mathcal{N}} P([x, w], (ie, 0)) |f([x_0, w_0] \cdot [Q(y^{1/2})x, Q(y^{1/2})w]) - f([x_0, w_0])| dx dw \\ &= \int_{\mathcal{N}} P([x - 2 \operatorname{Im} Q(y^{-1/2}) \Phi(w_0, Q(y^{1/2})w), w], (ie, 0)) \\ &\quad \times |f([x_0 + Q(y^{1/2})x, w_0 + Q(y^{1/2})w]) - f([x_0, w_0])| dx dw, \end{aligned}$$

where we have used the formula for group multiplication and changed variables in $\text{Re } V_1$.

Assume now that the set H^* is of the form $H \times K_2$, where K_2 is the cube about the origin in V_2 of side two. Notice that if $w, w_0 \in K_1$, the unit cube in V_2 , and if $|y| \leq \beta > 0$, then $w_0 + Q(y^{1/2})w \in K_2$. Moreover, if $w, w_0 \in K_1$, then

$$P([x - 2 \operatorname{Im} Q(y^{-1/2}) \Phi(w_0, Q(y^{1/2})w), w], (ie, 0))$$

is bounded below for $x \in Q_1$, the unit cube in $\operatorname{Re} V_1$, as $y \rightarrow 0$. Therefore, if $w_0 \in K_1$, we can bound Δ_y from below by integrating in (3) only over $Q_1 \times K_1$, obtaining:

$$\Delta_y \geq C \int_{Q_1} |\chi_H(x_0 + Q(y^{1/2})x) - \chi_H(x_0)| dx,$$

and we are reduced to the tube domain case. In particular, if $H \subset \operatorname{Re} V_1$ is the set which provides the counterexample for the tube domain corresponding to D , and if

$$\tilde{H} = \{(x, w) : x \in H, w \in K_1\},$$

then for all $u \in \tilde{H}$, $F(u_y) \nrightarrow f(u)$ as $y \rightarrow 0$ unrestrictedly. This completes the proof of Theorem 4.1(a).

REMARK. Since $F(u_y) \rightarrow f(u)$ a.e. as $y \rightarrow 0$ restrictedly, it is clear that $F(u_y)$ can have no limit as $y \rightarrow 0$ unrestrictedly.

Proof of Theorem 4.1(b). Let $p < \infty$. We show first that given any $\delta > 0$, there exists an $f = f_\delta \in L^p(B)$, so that $\sup_{y \in \Omega_\delta} F(u_y) = \infty$, for a.e. $u \in B$, where $\Omega_\delta = \Omega \cap \{|y| < \delta\}$. Assume in the contrary direction that $\sup_{y_n} |F(u_{y_n})| < \infty$ a.e. in u whenever $f \in L^p$, where $\{y_n\}$ is a fixed sequence dense in Ω_δ . However, if f is continuous and has compact support, $\lim_{y'_n \rightarrow 0} F(u_{y'_n})$ exists for every u , as long as $\{y'_n\}$ is a sequence that tends to zero. Thus by a well-known argument (see [DS, p. 333]), whenever $f \in L^p$, $\lim_{y_n \rightarrow 0} F(u_{y_n})$ exists almost everywhere, no matter what subsequence $\{y'_n\}$ of $\{y_n\}$ we pick, as long as $y'_n \rightarrow 0$. This is a contradiction with Theorem 4.1(a), when $f = \chi_H$.

Therefore there exists an $f_0 \in L^p(B)$ so that $\sup_{y \in \Omega_\delta} |F_0(u_y)| = \infty$ for a set u of positive measure. We assume, as we may, that $f_0 \geq 0$; and let $f = \sum 2^{-j} f_j$, with f_j the translate of f_0 by r_j . Then as in the proof of Theorem 4.1(a), $f \in L^p(B)$ and $\sup_{y \in \Omega_\delta} F(u_y) = \infty$ almost everywhere. Denote this f by f_δ , and assume that $\|f_\delta\|_p \leq 1$. Finally take $f = \sum_{j=1}^\infty 2^{-j} f_{\delta_j}$, where $\delta_j = 1/j$. Then for this f we have clearly $\limsup_{y \rightarrow 0} F(u_y) = \infty$, for almost every u .

CONCLUDING REMARKS. The following observation may help to put the result in better perspective. The analog of the Jessen-Marcinkiewicz-Zygmund result holds for any product of irreducible domains of the type we have considered. Specifically, if $D = D_1 \times \cdots \times D_k$ and if $y = (y_1, \dots, y_k) \rightarrow 0$ in such a way that each y_j stays within a subcone $\Omega'_j \subset \Omega_j$, then $F(u_y) \rightarrow f(u)$ a.e. on B if $f \in L^p(B)$, $p > 1$. This follows from repeated application of the norm estimates in $[W_1]$ and $[W_2]$.

On the other hand, in order for a.e. convergence to hold when f is merely integrable, it is necessary that the $|y_j|$ be comparable as they tend to 0, i.e., $a < |y_i|/|y_j| < a'$. That is, y must stay within a subcone of $\Omega_1 \times \cdots \times \Omega_k$.

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